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ON THE DETERMINATION OF THE RADIATION PATTERN OF AN EMITTER FRO--ETC(U)
SEP 60 G S INNIS, C W HORTON NOBSR-72627

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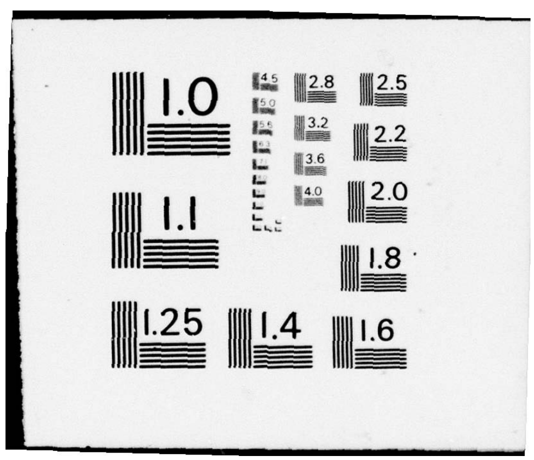
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ON THE DETERMINATION OF THE RADIATION PATTERN OF AN EMITTER FROM NEARFIELD DATA.

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by

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G. S. Innis, Jr. ~~and~~ C. W. Horton

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DRL Acoustical Report, No. 177

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Resulting from research done under Bureau of Ships
Research and Development Contract NObsr-72627
NE 051247-6, NE 051456-4
Problem 22-C
Sponsored jointly by the Bureau of Ships
and the Office of Naval Research, Code 411

107500 4B

DEFENSE RESEARCH LABORATORY
THE UNIVERSITY OF TEXAS
AUSTIN 12, TEXAS

POST OFFICE BOX 8029
TELETYPE AZ 79

800 EAST 24TH STREET
GREENWOOD 8-6612

ERRATA FOR DRL ACOUSTICAL REPORT NO. 177

Page 16, Equation (3.14)	The factors $(\xi_s^2+\eta_o^2-1)(1-\eta_o^2)^{1/2}$ should be replaced by $(\xi_s^2-\eta_o^2)^{1/2}$
Page 21, Equation (4.3)	The term $(\xi^2 -1)$ in the denominator of the right member should read $(\xi_s^2 -1)$.
Page 22, Equation (5.1)	The factors $(1 - \eta_o^2)^{1/2}(\xi_s^2 + \eta_o^2 - 1)$ should be replaced by unity and m should be replaced by zero.
Page 31	The expansions in Eq. (A.13) are really for $e^{ikR}/4\pi R$ rather than for e^{ikR}/R as stated.
Page 32	The statement in line 2 is wrong. One has

$$\frac{\partial G(\underline{r}|\underline{r}_o^s)}{\partial n} = \frac{1}{k_\xi} \left. \frac{\partial G(\underline{r}|\underline{r}_o^s)}{\partial \xi} \right|_{\xi=\xi_o}$$

ABSTRACT

In this report the problem of determining the field due to a distributed source is discussed. Only fields in which the disturbance is propagated in accordance with the classical, linear, scalar, homogeneous equation of wave motions are considered. The Green's function is used to derive two solution forms to the problem, and it is noted that these same forms have been discovered by other investigators using different techniques.

One of the forms presented requires the specification of inhomogeneous Cauchy boundary conditions ~~(ψ and $\partial\psi/\partial n$)~~ on the bounding surface while the other form requires only inhomogeneous Dirichlet conditions ~~(ψ)~~ on the boundary. In the method requiring Cauchy conditions, an approximate relation ~~between ψ and $\partial\psi/\partial n$~~ is found so that, in effect, Dirichlet conditions are specified. The error introduced by making this approximation is mentioned.

Finally, the radiation pattern of an underwater acoustic radiator is computed using each of the two methods and is compared with the measured radiation pattern.

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CHAPTER I

Introduction

The problem of computing the values of a function Ψ in a closed volume V in which Ψ satisfies the scalar wave equation from values of Ψ and/or $\partial\Psi/\partial n$ over the bounding surface S of V has been studied extensively[1,2,3,4] and there are apparently three different methods of solving the problem. They are

1. the Green's function approach in which the solution $\Psi(P)$ at the point P is expressed as an integral over the surface S of the normal derivative of the appropriate Green's function multiplied by the inhomogeneous Dirichlet boundary conditions. Alternately, the appropriate Green's function may be multiplied by the inhomogeneous Neumann boundary conditions and integrated over the surface S to compute the solution $\Psi(P)$. The use

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[1]Stuart Ballantine, "An Operational Proof of the Wave-Potential Theorem, with Applications to Electromagnetic and Acoustic Systems," The Journal of the Franklin Institute, Lancaster Press, Inc., Lancaster, Pennsylvania, April 1936, vol. 221, no. 4, pp. 469-484.

[2]Philip M. Morse and Herman Feshbach, Methods of Theoretical Physics, McGraw-Hill Book Company, Inc., New York, 1953, pp. 834-857.

[3]Jaroslav Pachner, "On the Dependence of Directivity Patterns on the Distance from the Emitter," Journal of the Acoustical Society of America, Lancaster, Pennsylvania, January 1956, vol. 28, no. 1, pp. 86-90.

[4]Bevin B. Baker and E. T. Copson, The Mathematical Theory of Huygens' Principle, Oxford at the Clarendon Press, London, England, 1950, pp. 23-27.

of inhomogeneous Dirichlet boundary conditions is preferred to the Neumann boundary conditions in many practical situations because a knowledge of the function $\Psi(S)$ on the surface S is more common than a knowledge of the normal gradient $\partial\Psi(S)/\partial n$ on the surface S.

2. the Kirchhoff-Helmholtz method in which the solution $\Psi(P)$ at the point P is expressed as an integral over the surface S of a function involving both the values of $\Psi(S)$ and of $\partial\Psi(S)/\partial n$ on the surface S. The obvious disadvantage here is that one is required to specify inhomogeneous Cauchy boundary conditions.

3. the modified Pachner method, so called because the basic idea was first published by J. Pachner.[5] This method requires the specification of inhomogeneous Dirichlet conditions over the surface S and expresses the solution $\Psi(P)$ at the point P in terms of a doubly infinite series.

It is the purpose of this paper to show that these three forms are not three different and independent solutions to the problem but that the second and third forms may be derived from the first. By using the Green's function for a point source in free space, one can derive the Kirchhoff-Helmholtz formula in which specification of inhomogeneous Cauchy boundary conditions is necessary. By choosing the Green's function which vanishes over the surface S, one can derive the modified Pachner method.

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[5] Pachner, op. cit., p. 86.

To illustrate the methods derived, the radiation pattern for an underwater acoustic radiator will be computed from measurements made in the nearfield. As the radiator to be considered is a long, thin, circular cylinder, the prolate spheroidal coordinate system will be used. This choice is made because a prolate spheroid can be made to fit a portion of a right circular cylinder as closely as one desires. Cylindrical coordinates are avoided because their use would necessitate the specification of the boundary values $\Psi(S)$ and/or $\partial\Psi(S)/\partial n$ over a surface of infinite extent. One would prefer to consider a surface S which is closed in a finite region of the space considered. One could, of course, choose the surface of the circular cylinder contained between two planes perpendicular to the axis of the cylinder; but then the surface of integration (the portion of the cylinder plus the two end caps) is not a coordinate surface, and the specification of the Green's function becomes exceedingly difficult. Hence, pressure measurements over the surface of a prolate spheroid in the nearfield of the acoustic radiator will be used to compute the pressure in the farfield (Fraunhoffer zone) of this radiator using both the Kirchhoff-Helmholtz solution and the modified Pachner solution. These computed values will then be compared with observed values.

CHAPTER II

The Kirchhoff-Helmholtz Formula

Small amplitude acoustic disturbances are propagated in many media in accordance with the classical scalar equation of wave motions, i.e.

$$\nabla^2 \Psi = \frac{1}{c^2} \frac{\partial^2 \Psi}{\partial t^2} . \tag{2.1}$$

Solutions of eq. (2.1) containing only the single harmonic angular frequency ω such that $\Psi = \psi e^{-i\omega t}$ are of particular interest. By restricting the discussion to disturbances of this type, eq. (2.1) may be written

$$\nabla^2 \psi + k^2 \psi = 0 . \tag{2.2}$$

Equation (2.2) is the well known scalar Helmholtz equation, and $\nabla^2 + k^2$ is often called the Helmholtz operator.

It can be shown[6] that the solution of eq. (2.2) can be written as

$$\frac{1}{4\pi} \oint \left[G(\underline{r}|\underline{r}_o^s) \frac{\partial \psi(\underline{r}_o^s)}{\partial n} - \psi(\underline{r}_o^s) \frac{\partial G(\underline{r}|\underline{r}_o^s)}{\partial n} \right] dA_o . \tag{2.3}$$

[6] Morse and Feshbach, *op. cit.*, pp. 804-806.

Expression (2.3) has the value of $\psi(\underline{r})$ or zero according as $\underline{r}$ lies interior or exterior to the bounding surface S [see Fig. (2.1)]. The superscript s denotes values taken on the surface S, and the subscript o denotes source point coordinates. n is the unit outward normal to the surface S.

The necessary Green's function in (2.3) is simply[7] that for the point source in free space

$$G(\underline{r}|\underline{r}_o^s) = \frac{e^{ikR}}{R} \tag{2.4}$$

in which $$R = |\underline{r} - \underline{r}_o^s| .$$

Using eq. (2.4) in (2.3), one gets for the region interior to the surface S

$$\psi(\underline{r}) = \frac{1}{4\pi} \oint \left[\frac{e^{ikR}}{R} \frac{\partial\psi(\underline{r}_o^s)}{\partial n} - \psi(\underline{r}_o^s) \frac{\partial}{\partial n} \left(\frac{e^{ikR}}{R} \right) \right] dA_o . \tag{2.5}$$

Let only those source distributions be considered for which there exists a surface σ, which completely encloses the sources of the field and such that, for all points p which lie on the surface σ, $|p| < Q$, where Q is some positive number. One would like an expression for the solution $\psi(\underline{r})$ in the region exterior to the closed surface σ . To determine a solution in this form, consider the doubly connected surface

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[7]*Ibid.*, p. 810.

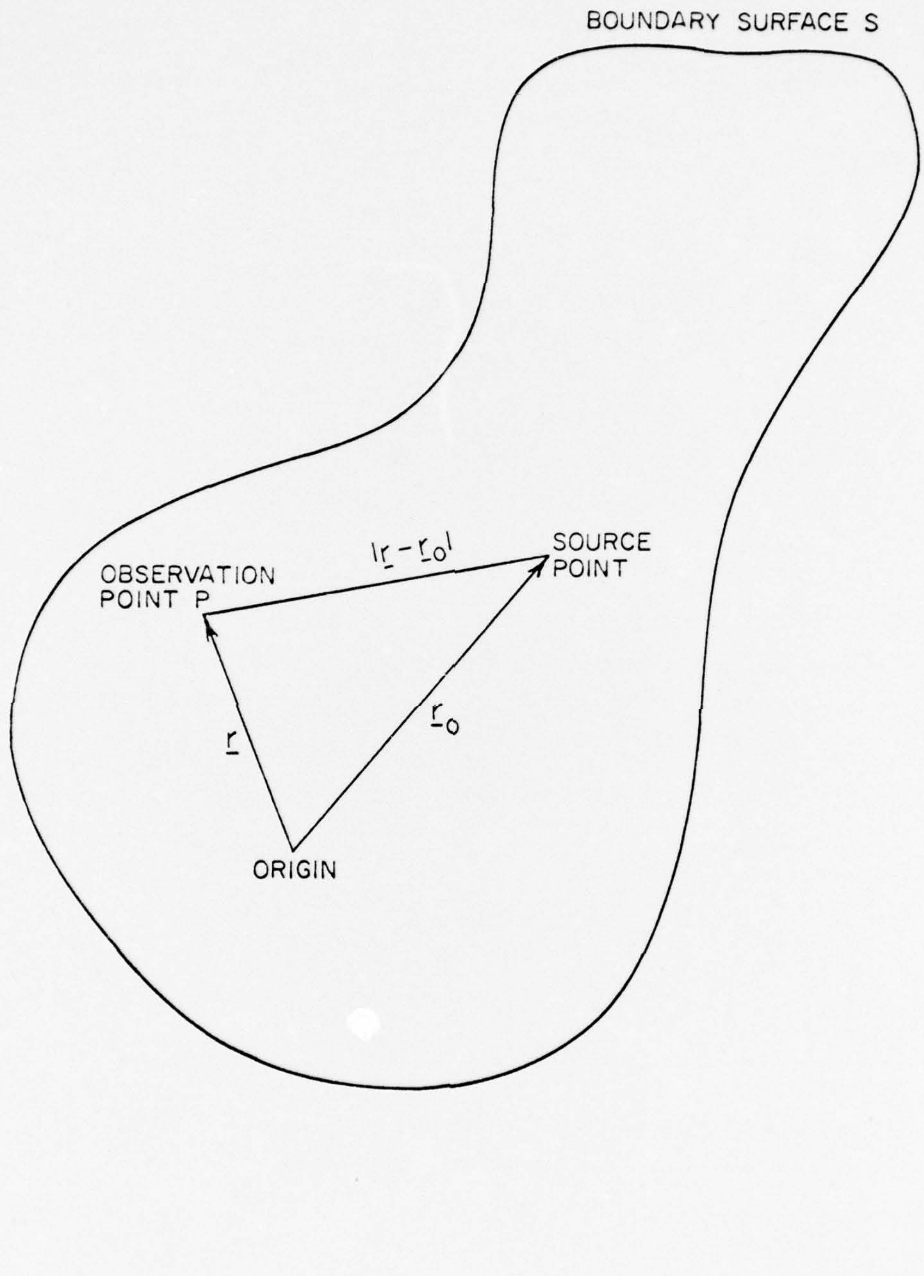

FIG. 2.1 - SOURCE POINT, OBSERVATION POINT AND BOUNDARY SURFACE FOR GREEN'S FUNCTION

$\sigma + \Sigma$ such that all the sources of the field lie in the volume V bounded externally by σ. Let Σ be a sphere of radius ρ and be centered on the origin. It is easily shown[8] that the contribution to eq. (2.5) from the integration over Σ as $\rho \to \infty$ is zero provided Sommerfield's radiation condition ("Ausstrahlungsbedingung") holds, i.e.

$$\rho \left[\frac{\partial \psi}{\partial \rho} - ik\psi \right] \to 0 \quad \text{as} \quad \rho \to \infty \tag{2.6}$$

uniformly with respect to the polar angles.

Equation (2.5) can thus be considered as giving the value of the function $\psi(\underline{r})$ in the region of space exterior to a given closed surface S (σ in the preceeding paragraph) in terms of the values of $\psi(\underline{r}_o^S)$ and the normal gradient $\partial\psi(\underline{r}_o^S)/\partial n$; however, n (the outward normal to $\sigma + \Sigma$) is the inward normal to the surface S. If we wish n to denote the outward normal to S, we write

$$\psi(\underline{r}) = \frac{1}{4\pi} \oint \left[\psi(S) \frac{\partial}{\partial n} \left(\frac{e^{ikR}}{R} \right) - \frac{e^{ikR}}{R} \frac{\partial \psi(S)}{\partial n} \right] dA \tag{2.7}$$

in which $\psi(S)$ is written for $\psi(\underline{r}_o^S)$ for simplicity. Equation (2.7) is the well known Kirchhoff-Helmholtz equation.[9]

The use of eq. (2.7) for the calculation of $\psi(\underline{r})$ is, in many cases, impractical because it requires a knowledge of Cauchy boundary conditions

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[8] Baker and Copson, op. cit., p. 25 (see footnotes).

[9] Ibid., p. 26.

which may not be known. In order to specify only the simpler Dirichlet boundary conditions, an expression that relates $\psi(S)$ to $\partial\psi(S)/\partial n$ is sought.

For simple harmonic disturbances Ψ, a vector $\underline{u}$ can be associated with ψ

$$\underline{u} = \frac{1}{i\omega\rho} \text{ grad } \psi \ . \tag{2.8}$$

In particular, if ψ is the acoustic pressure in a medium of average density ρ, $\underline{u}$ is the particle velocity. The normal component of the particle velocity is in magnitude

$$u_n = \frac{1}{i\omega\rho} \frac{\partial\psi}{\partial n} \ . \tag{2.9}$$

Unfortunately, measurement of the normal component of the velocity is not easy. However, if the direction of the particle motion is normal to the surface S, then

$$u = u_n = \frac{k}{\omega\rho} \psi \ . \tag{2.10}$$

Substituting from eq. (2.10) into eq. (2.9)

$$\frac{k}{\omega\rho} \psi = \frac{1}{i\omega\rho} \frac{\partial\psi}{\partial n} \tag{2.11}$$

or

$$\frac{\partial\psi}{\partial n} = ik\psi \qquad (2.12)$$

Thus a simple expression for the $\partial\psi/\partial n$ is obtained in terms of ψ assuming that the radiated waves are locally plane waves radiated in the direction of the outward normal to the surface S. Müller[10] has shown that the error in eq. (2.10) for a spherically spreading wave is of the order of $1/R^2$ where R is the radius of curvature of the wavefront.

Using eq. (2.12) in eq. (2.7), one obtains finally

$$\psi(\underline{r}) = \frac{1}{4\pi} \oint \left[\frac{\partial}{\partial n} \left(\frac{e^{ikR}}{R} \right) - ik \frac{e^{ikR}}{R} \right] \psi(S) \; dA \qquad (2.13)$$

which requires only the Dirichlet boundary values $\psi(S)$.

It should be noticed that the values of $\psi(S)$ specified over the surface S will, in general, be complex as will the computed values $\psi(\underline{r})$ at the field point $\underline{r}$.

[10] Claus Müller, "Electromagnetic Radiation Patterns and Sources," IRE Transactions on Antennas and Propagation, Professional Group on Antennas and Propagation, July 1956, vol. AP-4, no. 3, p. 229.

CHAPTER III

Application of Kirchhoff-Helmholtz Formula

The form of the solution of the Helmholtz eq. (2.2) in the prolate spheroidal coordinate system has been known since 1880. However, due to the complexity of this form, it is unfamiliar to many. Therefore Appendix A has been included in which the form of this solution and several of its properties are discussed in detail. The results obtained in Appendix A will be assumed in the remainder of the discussion.

The problem of notation in this coordinate system demands a word of discussion. As yet no notation has been agreed upon by the various workers in the field; for that matter, even the method of normalizing some of the functions is still a matter of discussion. Flammer,[11] however, lists and discusses the various notations so the details of these systems will not be mentioned here other than as pertains to the notation of Flammer which is used exclusively. Flammer's notation does not include a simple symbol for the expansion coefficient for the radial functions, but the obvious choice for this symbol has been made [see eq. (A.9) of Appendix A] and is used throughout this paper.

One would like to write eq. (2.13) in a form such that, having measured the values of the acoustic pressure $\psi(S)$ at discrete points on the surface S, one can compute the acoustic pressure at the field point $\underline{r}$.

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[11] Carson Flammer, Spheroidal Wave Functions, Stanford University Press, Stanford, California, 1957, pp. 14-15.

To this end consider

$$\frac{\partial}{\partial n}\left[\frac{e^{ikR}}{R}\right] = \left[-\frac{1}{R} + ik\right]\frac{e^{ikR}}{R}\frac{\partial R}{\partial n} \quad . \tag{3.1}$$

Now if the point $\underline{r}$ is a great distance from the nearest point of the surface S, i.e., $R \gg 1$, the first term on the right of eq. (3.1) can be neglected compared with the second term. On consideration of Fig. (3.1) the factor $\partial R/\partial n$ is easily seen to be

$$\frac{\partial R}{\partial n} = -\cos\theta \quad . \tag{3.2}$$

One finds thus

$$\frac{\partial}{\partial n}\left[\frac{e^{ikR}}{R}\right] \cong -ik\cos\theta\ \frac{e^{ikR}}{R} \tag{3.3}$$

for points $\underline{r}$ a great distance from the surface of integration. Now eq. (2.13) can be written

$$\psi(\underline{r}) \cong -\frac{ik}{4\pi}\oint(1+\cos\theta)\frac{e^{ikR}}{R}\ \psi(S)\ dA \quad . \tag{3.4}$$

At this point there are at least two paths which lead to a tractable numerical approximation to eq. (3.4). Both of these methods involve determining an expression for the cosine θ in terms of the field point and a general point on the surface $\xi_o = \xi_s$. As deriving this expression involves some detailed, though not difficult, geometric manipulation, the first part of Appendix B is devoted to this calculation. The choice of

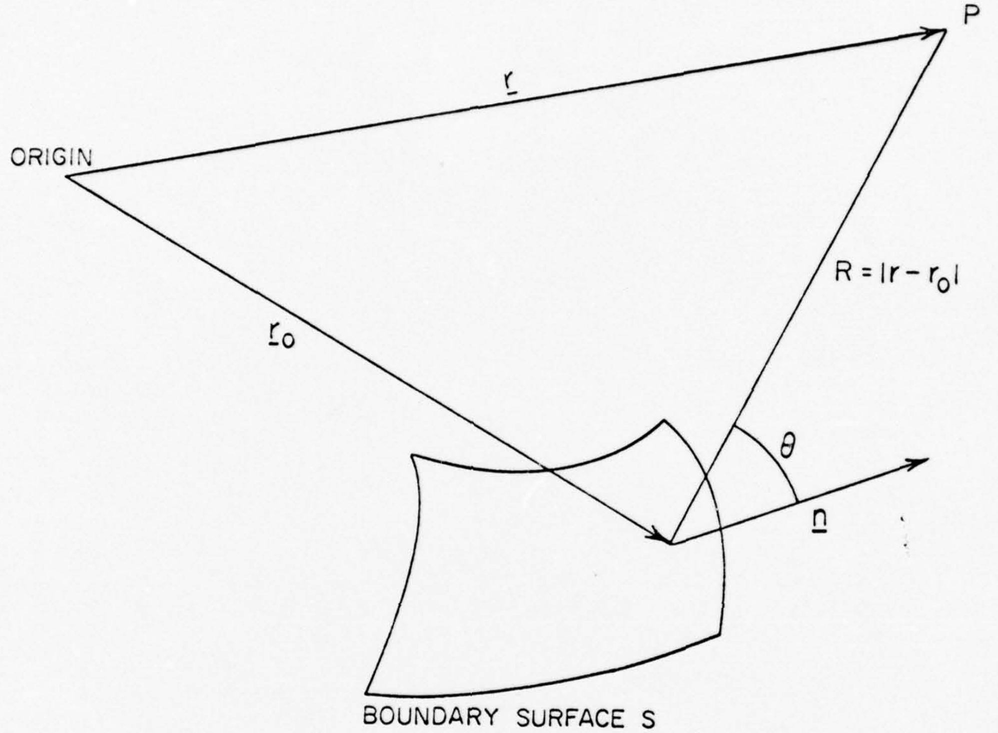

FIG. 3.1 – A PORTION OF THE SURFACE S SHOWING THE OUTWARD NORMAL $\underline{n}$, THE FIELD POINT $\underline{r}$, THE SOURCE POINT $\underline{r}_o$, THEIR DIFFERENCE R, AND THE ANGLE θ

approaches is due to the fact that there are probably several ways of computing (exp ikR)/R . Two of these methods, the use of an infinite series expansion in terms of the spheroidal wave functions and a conceptually simpler geometrical calculation, will be expounded separately.

First, the more sophisticated method shall be considered, i.e., an infinite series expansion of (exp ikR)/R shall be used in eq. (3.4) to compute the pressure at the field point $\underline{r}$. With eqs. (A.12) and (B.8) one can write immediately

$$\psi(\underline{r}) = \frac{k^2}{\pi^2} \sum_{m=0}^{\infty} \sum_{n=m}^{\infty} \frac{2-\delta_{om}}{N_{mn}(c)} S_{mn}(c,\eta) R_{mn}^{(3)}(c,\xi) R_{mn}^{(1)}(c,\xi_s)$$

$$\oint \left[1 + \frac{\eta\eta_o(\xi_s^2 - 1)^{\frac{1}{2}}}{(\xi_s^2 - \eta_o^2)^{\frac{1}{2}}} + \frac{\xi_s (1 - \eta^2)^{\frac{1}{2}} (1 - \eta_o^2)^{\frac{1}{2}}}{(\xi_s^2 - \eta_o^2)^{\frac{1}{2}}} \cos\varphi_o \right]$$

$$S_{mn}(c,\eta_o) \cos[m(\varphi - \varphi_o)] \; \psi(\eta_o, \xi_s, \varphi_o) \; dA \quad . \qquad (3.5)$$

in which the surface $\xi_o = \xi_s$ is the surface of integration.

There are several observations that can be made to simplify eq. (3.5). First, since the pressure $\psi(\underline{r})$ at a farfield point $\underline{r}$ is to be computed, an asymptotic expansion for $R_{mn}^{(3)}(c,\xi)$ can be used [see eq. (A.10) of Appendix A]. Secondly, the hydrophone used in these tests was constructed and operated in such a way that the field was unchanged by rotation of the hydrophone about its long axis (i.e., the field was omnidirectional in any plane perpendicular to the line determined by $\xi = 1$).

Consequently, the integration with respect to φ_o in eq. (3.5) can be performed analytically, and the field study can be restricted to the plane $\varphi = 0$ without loss of generality. Thirdly, during these tests, only relative amplitudes will be compared and, consequently, all constant factors can be ignored. These terms would, of course, be important in the determination of the absolute pressure in the farfield from nearfield measurements. Equation (3.5) then becomes

$$\psi(\underline{r}) \sim \sum_{m=0}^{1} \sum_{n=m}^{\infty} \frac{2 - \delta_{om}}{N_{mn}(c)} S_{mn}(c,\eta) \, e^{-i \frac{n+1}{2}\pi} R_{mn}^{(1)}(c,\xi_s)$$

$$\int_{-1}^{1} \left[\left(1 + \frac{\eta\eta_o(\xi_s^2 - 1)^{\frac{1}{2}}}{(\xi_s^2 - \eta_o^2)^{\frac{1}{2}}} \right) \delta_{om} + \left(\frac{\xi_s(1 - \eta^2)^{\frac{1}{2}} (1 - \eta_o^2)^{\frac{1}{2}}}{(\xi_o^2 - \eta_o^2)^{\frac{1}{2}}} \right) (1 - \delta_{om}) \right]$$

$$S_{mn}(c,\eta_o) \, \psi(\eta_o,\xi_s) \, (\xi_s^2 + \eta_o^2 - 1) \, (1 - \eta_o^2)^{\frac{1}{2}} \, d\eta_o \qquad (3.6)$$

and to compute $\psi(\underline{r})$ at the point $\underline{r}$ all that is necessary is that one measure $\psi(\eta_o,\xi_s)$ over the surface $\xi_o = \xi_s$ and apply eq. (3.6). Since there is no dependence on φ_o , measurements need only be made along the portion of the ellipse determined by $\xi_o = \xi_s$ and $\varphi_o = 0$.

As eq. (3.6) is obviously one that would be difficult to handle numerically, one considers the other possibility, i.e., the determination of $(\exp ikR)/R$ geometrically. The simpler of these two results will then be used in this application.

In Appendix B simple geometry is used to find an expression for $(\exp ikR)/R$ [see eq. (B.15)] in the prolate spheroidal coordinate system. These expressions will be used in eq. (3.4) to get an expression for $\psi(\underline{r})$ that is numerically tractable. To simplify the discussion define

$$E = -k\frac{d}{2}\ \eta\ \eta_o\ \xi_s = -c\ \eta\ \eta_o\ \xi_s \quad , \tag{3.7}$$

$$F = -c\ (1-\eta^2)^{\frac{1}{2}}\ [(1-\eta_o^2)\ (\xi_s^2-1)]^{\frac{1}{2}} \quad , \tag{3.8}$$

$$G = \frac{\eta\ \eta_o\ (\xi_s^2-1)^{\frac{1}{2}}}{(\xi_s^2-\eta_o^2)^{\frac{1}{2}}} \quad \text{and} \tag{3.9}$$

$$H = \frac{\xi_s\ (1-\eta^2)^{\frac{1}{2}}\ (1-\eta_o^2)^{\frac{1}{2}}}{(\xi_s^2-\eta_o^2)^{\frac{1}{2}}} \quad . \tag{3.10}$$

With the definitions (3.7) to (3.10) one has

$$\cos\ \theta = G + H\ \cos\ \varphi_o \tag{3.11}$$

and

$$\frac{e^{ikR}}{R} = k\ \frac{e^{ic\xi}}{c\xi}\ e^{iE + iF\cos\ \varphi_o} \tag{3.12}$$

(see Appendix B).

Placing these values in eq. (3.4) one has

$$\psi(\underline{r}) \cong -\frac{ik^2}{4\pi}\ \frac{e^{ic\xi}}{c\xi} \oint \Big[(1 + G + H\cos\varphi_o)\Big] e^{iE + iF\cos\varphi_o}\ \psi(\eta_o,\xi_s,\varphi_o)\ dA_o . \tag{3.13}$$

As mentioned earlier the discussion is restricted to relative values of both measured and computed quantities; therefore, constant factors can be ignored. Also, because the nearfield data do not depend on φ_o one can perform the integration with respect to φ_o [12] to get

$$\psi(\underline{r}) \sim \int_{-1}^{1} e^{iE} \left[(1+G)\, J_0\,(F) + iHJ_1\,(F)\right] (\xi_s^2+\eta_o^2-1)(1-\eta_o^2)^{\frac{1}{2}}\, \psi(\eta_o,\xi_s)\, d\eta_o \quad (3.14)$$

in which $J_0(F)$ and $J_1(F)$ are the ordinary Bessel functions of the first kind of orders 0 and 1 , respectively.

In spite of the algebraic complexity of the terms of eq. (3.14), defined by eqs. (3.7) to (3.10), it is not difficult to see that eq. (3.14) is far simpler than the corresponding eq. (3.6). The difficulty of computing spheroidal wave functions cannot be overemphasized. Even with modern high speed computers many procedural operations and numerical computations involved in using these functions remain unresolved. Simple expressions and simply evaluated expressions for these functions are as yet unknown. The limited tables of expansion coefficients of Stratton[13] *et al*. required, after two years of programming, ten hours of computing time on the Whirlwind I computer at the Massachusetts Institute of Technology. Consequently, in the remainder of this application of the Kirchhoff-Helmholtz formula, eq. (3.14) will be used.

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[12] N. W. McLachlan, *Bessel Functions for Engineers*, Oxford at the University Press, London, England, 1941, no. 30, p. 159.

[13] J. A. Stratton, P. M. Morse, L. J. Chu, J. D. C. Little, and F. J. Corbató, *Spheroidal Wave Functions*, John Wiley and Sons, Inc., New York, 1956.

Application of eq. (3.14) to a situation in which $\psi(\eta_o,\xi_s)$ is measured (or determined in some other way) for discrete values of η_o will, of course, require the numerical evaluation of the integral with respect to η_o . A thirteen point Gaussian Numerical Quadrature formula is used to evaluate the integral with respect to η_o . The thirteen point formula is chosen more or less arbitrarily. This number provides a reasonably good sampling of $\psi(\eta_o,\xi_s)$ on the surface of integration. The Gaussian formula[14] is used because it has been proven to give the best approximation to the integral for a given number of abscissae. This quadrature formula requires values of the integrand at points spaced in accordance with the zeros of the Legendre polynomials whose order is the number of abscissae to be considered (in this case 13).

For the problem considered here (the determination of the farfield radiation pattern from nearfield measurements) the integrand is determined in part by the experimentally measured values of the pressure $\psi(\eta_o,\xi_s)$ in the nearfield. Now, if one considers the variation of $\psi(\eta_o,\xi_s)$ with η_o ,one will find in general that $\psi(\eta_o,\xi_s)$ varies not only in amplitude but also in phase, i.e., the value of $\psi(\eta_o,\xi_s)$ at a point (η_o^1,ξ_s) will have associated with it an amplitude p_1 and a phase angle β_1 , whereas at the point (η_o^2,ξ_s) the amplitude will be p_2 and the phase angle will be β_2 . This combination of amplitude and phase angle suggest a complex

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[14] Lowan, Arnold N., Davids, Norman and Levenson, Arthur, "Table of the Zeros of the Legendre Polynomials of Order 1-16 and the Weight Coefficients of Gauss' Mechanical Quadrature Formula," Bulletin of the American Mathematical Society, vol. 28, no. 10, October 1942, pp. 739-743.

representation of the quantity $\psi(\eta_o, \xi_s)$ as $pe^{i\beta}$. It is in this sense that $\psi(\eta_o, \xi_s)$ is a complex quantity. Obviously, $\psi(\eta_o, \xi_s)$ written as $pe^{i\beta}$ is actually a complex representation of a real quantity. Because of the concern for a relative determination of $\psi(\underline{r})$, only relative values of p (to within a multiplicative constant) and of β (to within an additive constant) need be measured.

The values of p and β are measured at the points $\eta_o = 0.0$, ± 0.225, ± 0.454, ± 0.643, ± 0.799, ± 0.921, and ± 0.985 on the curve $\xi_s = 1.0125$ and $\varphi_o = 0$. These measured values are then used in eq. (3.14) with the integral numerically evaluated by Gaussian Numerical Quadrature to determine the computed radiation pattern shown in Fig. (3.2). The measured farfield pattern of Fig. (3.2) is the pattern for this hydrophone measured a great distance (30 feet) from the hydrophone.

The agreement between computed and measured values in Fig. (3.2) is seen to be quite good. Other tests of this method indicate that when the normal derivative of the pressure is given by ik times the pressure [as in eq. (2.12)], radiation patterns computed from eq. (3.4) are in excellent agreement with observed radiation patterns.

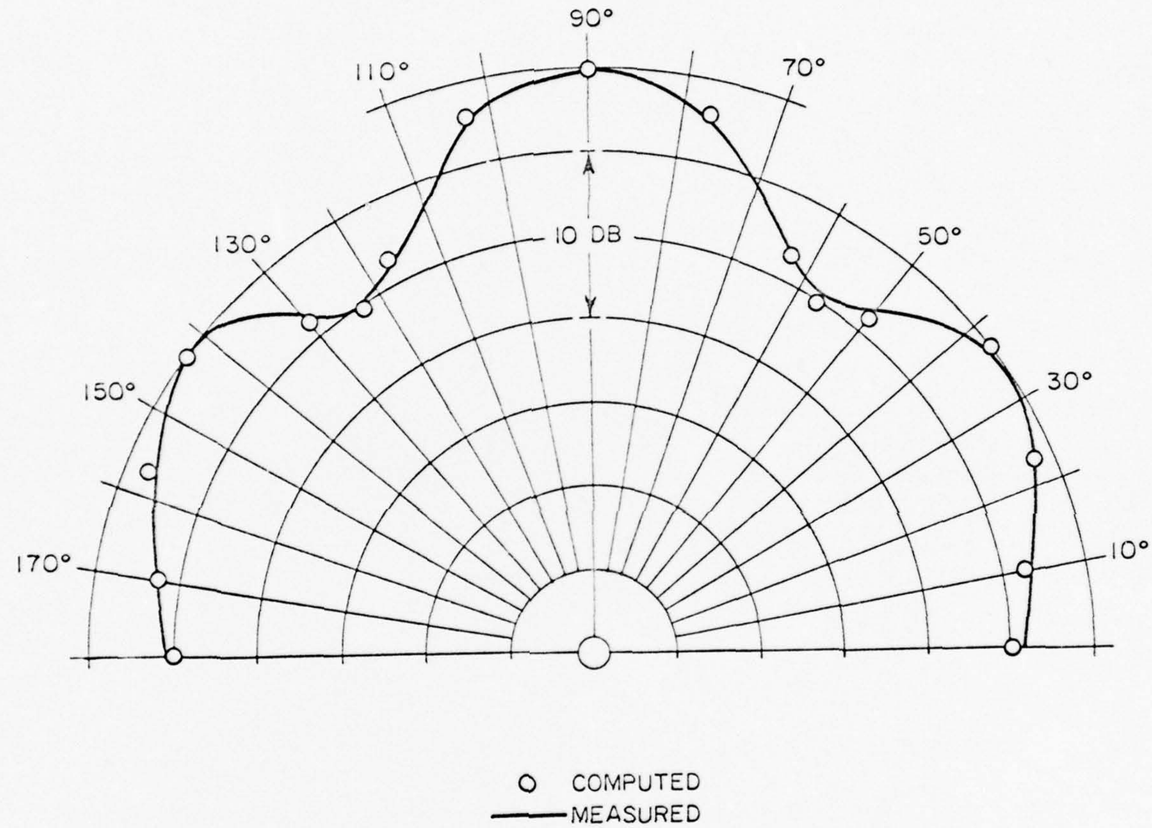

FIG. 3.2 - MEASURED AND COMPUTED RADIATION PATTERNS
KIRCHHOFF-HELMHOLTZ METHOD

CHAPTER IV

The Method of Pachner

A portion of Chapter II was taken up with the problem of determining a suitable approximation for the normal derivative of the function $\psi(\underline{r}_o^s)$. However, the Green's function in eq. (2.3) could be chosen to vanish over the surface of integration in which case the first term in the integrand of eq. (2.3) would vanish. One could, then, write

$$\psi(\underline{r}) = -\frac{1}{4\pi} \oint \left[\psi(\underline{r}_o^s) \quad \frac{\partial}{\partial n} G(\underline{r}|\underline{r}_o^s) \right] dA_o \tag{4.1}$$

in which $G(\underline{r}|\underline{r}_o^s)$ is now the Green's function which vanishes over the surface of integration S. Equation (4.1) is valid only when $\underline{r}$ lies within the surface S. As before, only the region exterior to the surface S is considered. It can be shown that

$$\psi(\underline{r}) = \frac{1}{4\pi} \oint \left[\psi(\underline{r}_o^s) \quad \frac{\partial}{\partial n} G(\underline{r}|\underline{r}_o^s) \right] dA_o \tag{4.2}$$

in which $\underline{r}$ is now exterior to the surface S and n is the unit outward normal to the surface S.

With the $\partial G(\underline{r}|\underline{r}_o^s)/\partial n$ of Appendix A [see eq. (A.15)], one can write immediately

$$\psi(\underline{r}) = -\frac{ik}{4\pi} \frac{1}{c(\xi^2-1)} \sum_{m=0}^{\infty} \sum_{n=m}^{\infty} \frac{2-\delta_{om}}{N_{mn}(c)} S_{mn}(c,\eta) \frac{R_{mn}^{(3)}(c,\xi)}{R_{mn}^{(3)}(c,\xi_s)}$$

$$\oint S_{mn}(c,\eta_o) \cos [m(\varphi-\varphi_o)] \quad \psi(\eta_o,\xi_s,\varphi_o) \quad dA_o \tag{4.3}$$

for the surface of integration $\xi_o = \xi_s$. Equation (4.3) is similar to Pachner's result. The exact result of Pachner can be derived by using the Green's function which vanishes over the surface of the sphere and by choosing the surface of integration to be the sphere. His result could also be derived from eq. (4.3) by considering the limiting case as the separation of the foci of the prolate spheroidal coordinate system goes to zero. This limiting process yields the spherical coordinate system, and the various spheroidal wave functions become spherical wave functions in the limit. Thus, eq. (4.3) contains Pachner's result as a special case.

Equation (4.3) is a significantly better method of computing $\psi(\underline{r})$ in many cases because the difficulty associated with approximating Cauchy boundary conditions with Dirichlet boundary conditions is avoided. No knowledge of the Neumann boundary conditions other than finiteness is required in the application of eq. (4.3) and, of course, finiteness is no essential restriction in many physical problems.

CHAPTER V

Application of the Method of Pachner

In this chapter the radiation pattern of the hydrophone considered in Chapter III will be computed using the method of Chapter IV. To simplify eq. (4.3), the observations mentioned earlier will be made. Specifically, the pressure distribution in the near and farfields does not depend on the coordinate φ. The point $\underline{r}$ at which $\psi(\underline{r})$ will be computed is a great distance from the surface of integration and only relative pressures will be measured and computed; thus, all constant factors can be ignored. With the use of these three observations, eq. (4.3) can be written

$$\psi(\underline{r}) \sim \sum_{n=0}^{\infty} \frac{S_{mn}(c,\eta)}{N_{mn}} \frac{e^{-i\frac{n+1}{2}\pi}}{R_{mn}^{(3)}(c,\xi_s)} \tag{5.1}$$

$$\int_{-1}^{1} \psi(\eta_o,\xi_s)\, S_{mn}(c,\eta_o)\, (1-\eta_o^2)^{\frac{1}{2}}(\xi_s^2+\eta_o^2-1)\, d\eta_o \quad .$$

The use of the thirteen point Gaussian Numerical Quadrature formula to compute the integral in eq. (5.1) then yields a numerically tractable expression. All of the values of the wave functions needed in this analysis are computed from the formulas of Appendix A. The determination of the necessary wave functions was not overly difficult because only the series on n occurs in eq. (5.1).

The computed and measured Fraunhoffer radiation patterns are shown in Fig. (5.1), and the agreement is again seen to be quite good.

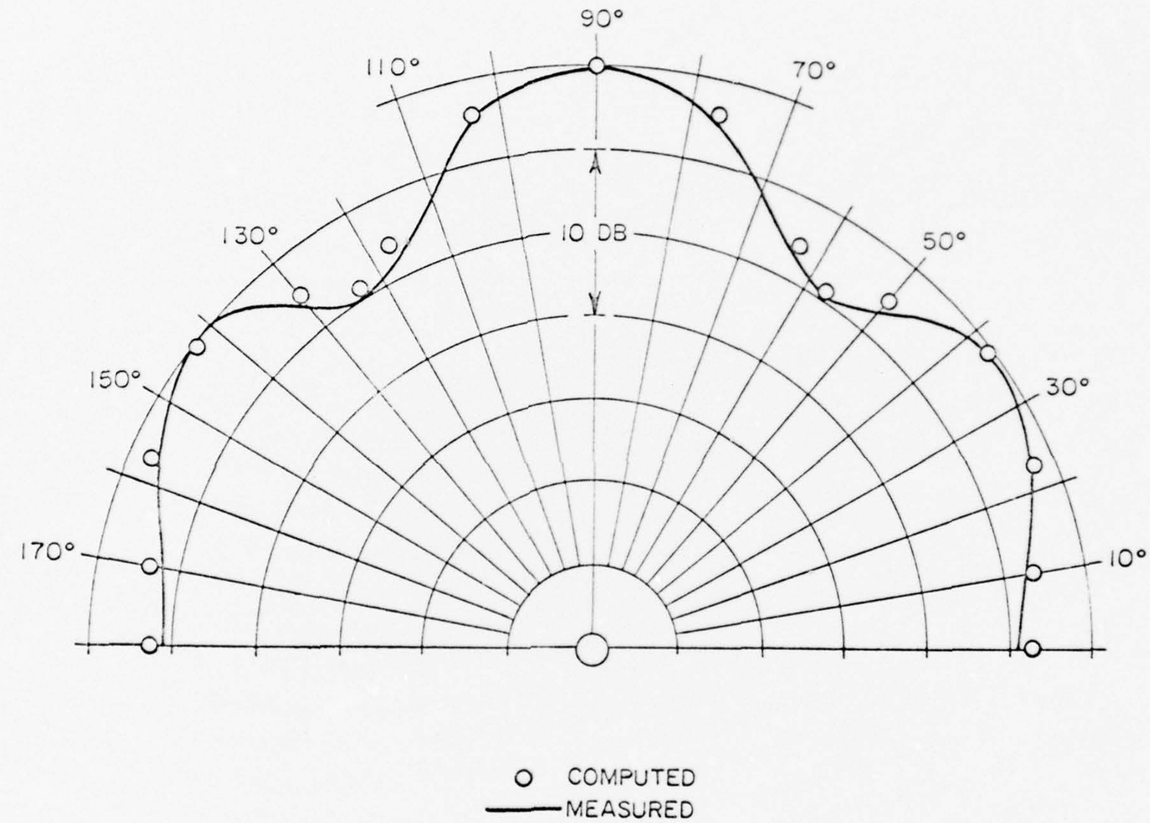

FIG. 5.1 - MEASURED AND COMPUTED RADIATION PATTERNS
METHOD OF PACHNER

CHAPTER VI

Conclusions

From Figs.(3.2) and (5.1) one can see that both the Kirchhoff-Helmholtz formula and the modified method of Pachner give very good agreement with observed values in the particular example considered here. Both methods have obviously good and bad points. The Kirchhoff-Helmholtz formula can be evaluated in such a way that one avoids the determination of the wave functions, but this formula also requires that one specify Cauchy boundary conditions even if only by approximation as in the example. The author has attempted to apply this formula in cases in which replacing Cauchy boundary conditions by Dirichlet boundary conditions via eq. (2.12) was not valid; and, as expected, the computed and measured radiation patterns did not agree.

The method of Pachner requires only the specification of Dirichlet boundary values but will in many cases raise more questions than it answers. The problem of computing the necessary wave functions is a difficult one. The rapidity of convergence of the series, in general, and the dependence of this convergence on the values of ξ_s and ξ are unknown.

In conclusion, the author's purpose is accomplished. There are not several basically different solutions to the problem of determining the values of a function which satisfies the Helmholtz equation exterior to a given surface from values on this surface, but there are merely several forms of one solution. Both of the forms of the solution have their good and their bad characteristics. The form to be used in any particular situation is

determined by what is known. In the event Cauchy boundary conditions are known, the Kirchhoff-Helmholtz approach is probably best; if Dirichlet conditions are known, the method of Pachner is preferred; whereas, if Neumann conditions are specified, one could find another form of the solution using the methods of Chapter IV by choosing the Green's function whose normal gradient vanishes over the boundary surface.

APPENDIX A

Speroidal Wave Functions

The prolate spheroidal coordinate system[15] is related to the rectangular Cartesian coordinate system via the system of equations

$$x = \frac{d}{2} \left[(1 - \eta^2)(\xi^2 - 1) \right]^{\frac{1}{2}} \cos\varphi \qquad \text{(A.1.a)}$$

$$y = \frac{d}{2} \left[(1 - \eta^2)(\xi^2 - 1) \right]^{\frac{1}{2}} \sin\varphi \qquad \text{(A.1.b)}$$

$$z = \frac{d}{2} \eta \xi \qquad \text{(A.1.c)}$$

with

$$-1 \leqslant \eta \leqslant 1 \qquad 1 \leqslant \xi < \infty \qquad 0 \leqslant \varphi \leqslant 2\pi$$

in which d is the separation of foci in the coordinate system. (see Note)

The Helmholtz equation, eq. (2.2), is separable in this coordinate system, giving the two ordinary differential equations

$$\frac{d}{d\eta}\left[(1-\eta^2)\frac{d}{d\eta} S_{mn}(c,\eta)\right] + \left[\lambda_{mn} - c^2\eta^2 - \frac{m^2}{1-\eta^2}\right] S_{mn}(c,\eta) = 0 \qquad \text{(A.2)}$$

and

$$\frac{d}{d\xi}\left[(\xi^2-1)\frac{d}{d\xi} R_{mn}(c,\xi)\right] - \left[\lambda_{mn} - c^2\xi^2 + \frac{m^2}{\xi^2-1}\right] R_{mn}(c,\xi) = 0 \qquad \text{(A.3)}$$

---

[15] Flammer, op. cit., p.6

Note: All square roots will be taken as non-negative.

in which the m and λ_{mn} are separation parameters and $c = \frac{1}{2}kd$ with k the wave number. The quantities $S_{mn}(c,\eta)$ are the prolate spheroidal angle functions, and the $R_{mn}(c,\xi)$ are the prolate spheroidal radial functions. Notice that eqs. (A.2) and (A.3) are, with a change of sign, identical. They are written separately only to emphasize the difference in the range of the two variables ξ and η. It should also be noted that eqs. (A.2) and (A.3) are of the form

$$\frac{d}{dz}\left[(1 - z^2)\frac{du}{dz}\right] + \left[\lambda - c^2z^2 - \frac{\mu^2}{1-z^2}\right] u = 0 \qquad \text{(A.4)}$$

which is the well known Sturm-Liouville equation. All of the theory of the Sturm-Liouville equation is thus applicable to eqs. (A.2) and (A.3); and, in particular, the orthogonality of the solutions u in eq. (A.4) will be of interest later in the development.

<u>The Prolate Spheroidal Angle Functions</u>

Equation (A.2), a second order ordinary differential equation, has two linearly independent solutions; however, one of these solutions, the angle function of the second kind $S^{(2)}_{mn}(c,\eta)$, is not finite for $\eta = \pm 1$. As these values are in the region of interest in this particular problem, the functions of the second kind will not be included in the formulation of the general solution and will not be considered hereafter. The other solution to eq. (A.2), the angle function of the first kind $S^{(1)}_{mn}(c,\eta)$, can thus be denoted by $S_{mn}(c,\eta)$ without ambiguity.

The solution to eq. (A.2) is determined by noticing that for $c = 0$, this differential equation is the one satisfied by the associated Legendre

functions of the first kind of integral order and degree. This suggests for the angle function of the first kind an infinite series of the form

$$S_{mn}(c,\eta) = \sum_{r=0,1}^{\infty}{}' \; d_r^{mn}(c) \quad P_{m+r}^{m}(\eta) \tag{A.5}$$

in which the prime over the summation sign indicates summation over only even values of r when $n-m$ is even and over only odd values of r when $n-m$ is odd. The expansion coefficients $d_r^{mn}(c)$ are determined by substitution of eq. (A.5) into eq. (A.2). Flammer[16] gives a recursion relation for the $d_r^{mn}(c)$ and also finds an expansion for the eigenvalues $\lambda_{mn}(c)$. Fortunately, however, the necessary expansion coefficients have been tabulated by Stratton, Morse, Chu, Little and Corbató.[17] The limitation of these tables to values of $c \leq 8$ is responsible for the choice of parameters (specifically frequency and length of radiator) used in the example considered in this thesis. With the tables of Stratton *et al.* and tables of associated Legendre functions, one can easily compute the angle functions of the first kind from eq. (A.5).

Orthogonality of the Functions $S_{mn}(c,\eta)$

From the general theory of Sturm-Liouville differential equations it follows that the functions $S_{mn}(c,\eta)$ are orthogonal in the interval $(-1,1)$. Therefore,

$$\int_{-1}^{1} S_{mn}(c,\eta) \; S_{mn'}(c,\eta) \; d\eta = \delta_{nn'} \, N_{mn} \tag{A.6}$$

---

[16] *Ibid.*, p. 17.

[17] Stratton *et al.*, *op. cit.*

in which the N_{mn} is easily determined by using the normalization factor for the associated Legendre function to be

$$N_{mn} = 2 \sum_{r=0,1}^{\infty}{}' \frac{(r + 2m)! \ [d_r^{mn}(c)]^2}{(2r + 2m + 1)\ r!} \quad . \qquad (A.7)$$

The Prolate Spheroidal Radial Functions

As in the case of the angle functions, there are two solutions to the differential equation satisfied by the radial functions; and one of these solutions is singular on the coordinate surface $\xi = 1$. Since, however, this singularity does not occur in the region of interest for this particular problem, both radial functions [the radial functions of the first kind $R_{mn}^{(1)}(c,\xi)$ and the radial function of the second kind $R_{mn}^{(2)}(c,\xi)$] will be included in the general solution. It is expedient to consider the radial function of the third kind given by

$$R_{mn}^{(3)}(c,\xi) = R_{mn}^{(1)}(c,\xi) + iR_{mn}^{(2)}(c,\xi) \quad . \qquad (A.8)$$

Flammer expands the function $R_{mn}^{(3)}(c,\xi)$ in terms of the expansion coefficients $d_r^{mn}(c)$ whereas Stratton <u>et al</u>. introduces a new expansion coefficient which in the notation of Flammer would be $a_r^{mn}(c)$ and which can, of course, be written in terms of $d_r^{mn}(c)$. Thus

$$R_{mn}^{(3)}(c,\xi) = \left[\frac{\xi^2 - 1}{\xi^2}\right]^{\frac{m}{2}} \sum_{r=0,1}^{\infty}{}' a_r^{mn}(c)\, h_{m+r}^{(1)}(c\xi) \qquad (A.9)$$

in which $h^{(1)}_{m+r}(c\xi)$ is the spherical Hankel function of the first kind. The expansion coefficients $a^{mn}_r(c)$ are chosen so that

$$R^{(3)}_{mn}(c,\xi) \xrightarrow[c\xi\to\infty]{} \frac{1}{c\xi}\, e^{\,i[c\xi-\frac{1}{2}(n+1)\pi]} \quad . \tag{A.10}$$

Expansion (A.9) is an asymptotic series being valid only when the significant values of $R^{(3)}_{mn}(c,\xi)$ lie in the range $0 < r < c\xi$. Because of the value of $c\xi$ ($c\xi = 8.1$) used in the examples worked out here, another expansion must be devised for the imaginary part of $R^{(3)}_{mn}(c,\xi)$. This is necessary because the significant terms in the expansion of $R^{(2)}_{mn}(c,\xi)$ do not occur in the range $0 < r < c\xi$. Flammer[18] derives the expression

$$R^{(2)}_{mn}(c,\xi) = \tfrac{1}{2}\, Q_{mn}\; R^{(1)}_{mn}(c,\xi)\; \log \frac{\xi+1}{\xi-1} + g_{mn}(c,\xi) \tag{A.11}$$

and gives details for computing the Q_{mn} [19] and g_{mn} .[20] $R^{(1)}_{mn}(c,\xi)$ is calculated in this case by replacing the spherical Hankel function in eq. (A.9) by the spherical Bessel function. The radial function of the third kind $R^{(3)}_{mn}(c,\xi)$ is then computed from eq. (A.8).

Green's Function in Prolate Spheroidal Coordinates

In Chapter II the simple Green's function (exp ikR)/R in prolate spheroidal coordinates is required. Flammer[21] writes this as

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[18] Flammer, op. cit., p. 35.

[19] Ibid., p. 35, eq. (4.4.6).

[20] Ibid., p. 35, eq. (4.4.7).

[21] Ibid., p. 47.

$$\frac{e^{ikR}}{R} = 2ik \sum_{m=0}^{\infty} \sum_{n=m}^{\infty} \frac{2-\delta_{om}}{N_{mn}(c)} \; S_{mn}(c,\eta)\, S_{mn}(c,\eta_o) \cos[m(\varphi-\varphi_o)]$$

$$\begin{cases} R_{mn}^{(1)}(c,\xi)\, R_{mn}^{(3)}(c,\xi_o) & \xi < \xi_o \\ R_{mn}^{(1)}(c,\xi_o)\, R_{mn}^{(3)}(c,\xi) & \xi > \xi_o \end{cases} \tag{A.12}$$

in which all terms [such as $N_{mn}(c)$, $S_{mn}(c,\eta)$ etc.] are exactly as defined earlier. The subscript o again is used to denote values taken in the source coordinates. As only the region exterior to the closed surface $\xi_o = \xi_s$ is considered, the lower term of eq. (A.12) will be used.

In Chapter IV, however, the Green's function which is $(\exp ikR)/R$ plus a general outgoing wave such that the sum satisfies the prescribed boundary conditions is of interest. For the problem considered herein the Green's function which vanishes over the surface of integration will be chosen so that only inhomogeneous Dirichlet boundary conditions need be specified for the problem. One gets, then,[22]

$$G(\underline{r}|\underline{r}_o^s) = \frac{e^{ikR}}{R} + \text{a general outgoing wave}$$

$$G(\underline{r}|\underline{r}_o^s) = \frac{ik}{2\pi} \sum_{m=0}^{\infty} \sum_{n=m}^{\infty} \frac{2-\delta_{om}}{N_{mn}} S_{mn}(c,\eta)\, S_{mn}(c,\eta_o) \cos[m(\varphi-\varphi_o)]$$

$$\left[R_{mn}^{(1)}(c,\xi_o)\, R_{mn}^{(3)}(c,\xi) - \frac{R_{mn}^{(1)}(c,\xi_s)}{R_{mn}^{(3)}(c,\xi_s)} R_{mn}^{(3)}(c,\xi_o)\, R_{mn}^{(3)}(c,\xi) \right] \tag{A.13}$$

for $\xi > \xi_o$.

---

22 Ibid., p. 47, eq. (5.2.11)-notice that $\cos[m(\varphi-\varphi_o)]$ is missing from eq. (5.2.11).

One will recall that the use of this function $G(\underline{r}|\underline{r}_o^s)$ in the text calls for $\partial G(\underline{r}|\underline{r}_o^s)/\partial n$. Obviously, $\partial G(\underline{r}|\underline{r}_o^s)/\partial n$ is $\partial G(\underline{r}|\underline{r}_o^s)/\partial \xi_o$ and the only factors to be differentiated are those contained in the braces. Consider, therefore,

$$\frac{\partial}{\partial \xi_o}\left[R_{mn}^{(1)}(c,\xi_o)\, R_{mn}^{(3)}(c,\xi) \quad - \quad \frac{R_{mn}^{(1)}(c,\xi_s)}{R_{mn}^{(3)}(c,\xi_s)}\, R_{mn}^{(3)}(c,\xi_o)\, R_{mn}^{(3)}(c,\xi)\right]$$

which is easily evaluated using the Wronskian for $R_{mn}^{(1)}(c,\xi)$ and $R_{mn}^{(2)}(c,\xi)$

$$R_{mn}^{(1)}(c,\xi)\ \frac{\partial R_{mn}^{(2)}(c,\xi)}{\partial \xi} - R_{mn}^{(2)}(c,\xi)\ \frac{\partial R_{mn}^{(1)}(c,\xi)}{\partial \xi} = \frac{1}{c(\xi^2-1)} \quad . \tag{A.14}$$

Using eq. (A.14) one gets for $\partial G(\underline{r}|\underline{r}_o^s)/\partial n$ [or $\partial G(\underline{r}|\underline{r}_o^s)/\partial \xi_o$] on the surface $\xi_o = \xi_s$

$$\left.\frac{\partial G(\underline{r}|\underline{r}_o^s)}{\partial \xi_o}\right|_{\xi_o=\xi_s} = -\ \frac{ik}{2\pi}\ \frac{1}{c(\xi^2-1)} \sum_{m=o}^{\infty}\sum_{n=m}^{\infty} \frac{2-\delta_{om}}{N_{mn}}\ S_{mn}(c,\eta)\ S_{mn}(c,\eta_o)\ \cos\,[m(\varphi-\varphi_o)]\ \frac{R_{mn}^{(3)}(c,\xi)}{R_{mn}^{(3)}(c,\xi_s)} \quad . \tag{A.15}$$

Thus, eq. (A.13) represents the Green's function which satisfies homogeneous Dirichlet boundary conditions on the surface, and eq. (A.15) is the normal derivative of this function.

APPENDIX B

Cosine θ

To compute the cosine of the angle between the normal to the surface S and the direction to the field point $(\xi,\eta,0)$, consider the equation for an ellipse in the x,z plane ($\varphi = 0$ plane). This equation can be solved for x giving

$$x = \pm \frac{b}{a} (a^2 - z^2)^{\frac{1}{2}} \tag{B.1}$$

and

$$\frac{dx}{dz} = \mp \frac{b}{a} \frac{z}{(a^2 - z^2)^{\frac{1}{2}}} \tag{B.2}$$

and it is well known that dx/dz is the slope of the tangent to the ellipse. It is also true from elementary analytic geometry that if m is the slope of a line and $m \neq 0$, then $-1/m$ is the slope of the perpendicular to this line. Hence, the slope of the normal to the ellipse is given by

$$\pm \frac{a}{b} \frac{(a^2 - z^2)^{\frac{1}{2}}}{z} = \tan \alpha_o$$

in which α_o is the angle the normal makes with the z axis.

Consider for the moment the quadrant in which both x and z are positive. In this quadrant the slope of the normal to the ellipse is positive everywhere and one can write

$$\cos \alpha_o = \frac{bz}{[z^2(b^2-a^2) + a^4]^{\frac{1}{2}}} \tag{B.3}$$

and

$$\sin \alpha_o = \frac{a\,(a^2-z^2)^{\frac{1}{2}}}{[z^2(b^2-a^2) + a^4]^{\frac{1}{2}}} \quad . \tag{B.4}$$

Now the expression for the $\cos \alpha_o$ is obviously valid for all x, z on the ellipse; however, the expression for the $\sin \alpha_o$ is valid only in the upper half plane $(x > 0)$ and must be replaced by its negative in the lower half plane $(x < 0)$.

In general, the plane containing the normal to the surface of a prolate ellipsoid and the major axis of the ellipsoid makes an angle φ with the x,z plane. Referring to Fig. (B.1) one sees that the angle between n and R is the angle θ; the angle between n and z is α_o; the angle between R and z is α; finally, the angle between the plane of R and z and the plane of n and z is φ_o. One can, therefore, compute the cosine of the angle θ from the well known relation for spherical triangles

$$\cos \theta = \cos \alpha_o \cos \alpha + \sin \alpha_o \sin \alpha \cos \varphi_o \quad . \tag{B.5}$$

It remains merely to observe that in the prolate spheroidal coordinate system the semi-major axis of the ellipse a and the semi-minor axis b are given by

$$a = \frac{d}{2}\,\xi_s \tag{B.6}$$

and

$$b = \frac{d}{2}\,(\xi_s^2 - 1)^{\frac{1}{2}} \quad . \tag{B.7}$$

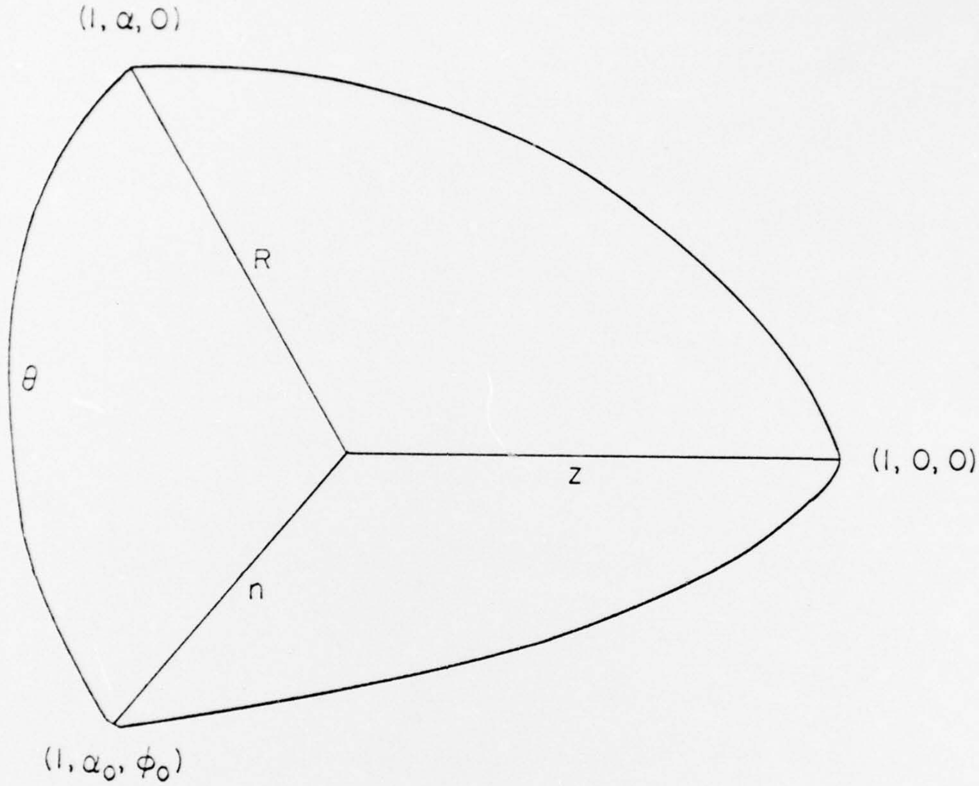

FIG. B.1 – THE SPHERICAL TRIANGLE USED TO DETERMINE COS θ

Using these expressions, one has

$$\cos\ \theta = \frac{\eta\eta_o(\xi_s^2-1)^{\frac{1}{2}}}{(\xi_s^2-\eta_o^2)^{\frac{1}{2}}} + \frac{\xi_s(1-\eta^2)^{\frac{1}{2}}\ (1-\eta_o^2)^{\frac{1}{2}}}{(\xi_s^2-\eta_o^2)^{\frac{1}{2}}}\ \cos\ \varphi_o \qquad \text{(B.8)}$$

in which the subscripts o denote values of the source coordinates, and the unsubscripted variables denote values at the field point. Notice that the $\cos\ \varphi_o$ term takes care of the change in sign for $\sin\ \alpha_o$ mentioned earlier. Thus, the angle between the normal at the point $(\eta_o, \xi_s, \varphi_o)$ to the prolate ellipsoidal surface $\xi_o = \xi_s$ and the line joining a farfield point $(\eta, \xi, 0)$ is given by eq. (B.8).

<u>(Exp ikR)/R</u>

To determine R one can write simply

$$R = \left[(x - x_o)^2 + (y - y_o)^2 + (z - z_o)^2\right]^{\frac{1}{2}} \qquad \text{(B.9)}$$

and use eqs. (A.1) to get R in prolate spheroidal coordinates. As in the calculation of cosine θ , the discussion will be restricted to field points $\underline{r}$ which lie in the plane $\varphi = 0$. Consequently, one gets for use in eq. (B.9)

$$x = \frac{d}{2}\left[(1 - \eta^2)\ (\xi^2 - 1)\right]^{\frac{1}{2}}$$

$$y = 0 \qquad \text{(B.10)}$$

$$z = \frac{d}{2}\ \eta\ \xi$$

and

$$x_o = \frac{d}{2} \left[(1 - \eta_o^2) \ (\xi_s^2 - 1) \right]^{\frac{1}{2}} \cos \ \varphi_o$$

$$y_o = \frac{d}{2} \left[(1 - \eta_o^2) \ (\xi_s^2 - 1) \right]^{\frac{1}{2}} \sin \ \varphi_o \qquad (B.11)$$

$$z_o = \frac{d}{2} \quad \eta_o \ \xi_s \qquad .$$

After algebraic manipulation, one gets

$$R^2 = \frac{d^2}{4} \left\{ (1-\eta^2) \ (\xi^2-1) + (1-\eta_o^2) \ (\xi_s^2-1) + \eta^2 \ \xi^2 + \eta_o^2 \ \xi_s^2 - 2\eta\eta_o\xi\xi_s \right.$$

$$\left. - 2 \left[(1-\eta^2) \ (\xi^2-1) \right]^{\frac{1}{2}} \left[(1-\eta_o^2) \ (\xi_s^2-1) \right]^{\frac{1}{2}} \cos \ \varphi_o \right\} \qquad (B.12)$$

and, on limiting the discussion to values of $\xi \gg 1$, one can write

$$R^2 \cong \frac{d^2}{4} \ \xi^2 \left[1 - \frac{2\eta\eta_o\xi_s}{\xi} - \frac{2(1-\eta^2)^{\frac{1}{2}} \ [(1-\eta_o^2) \ (\xi_s^2-1)]^{\frac{1}{2}} \ \cos \ \varphi_o}{\xi} \right] . \ (B.13)$$

By using the well known binomial theorem to approximate the square root on the right of eq. (B.13), one finds

$$R \cong \frac{d}{2} \left[\xi - \eta\eta_o\xi_s - (1-\eta^2)^{\frac{1}{2}} \ [(1-\eta_o^2) \ (\xi_s^2-1)]^{\frac{1}{2}} \ \cos \ \varphi_o + O(\tfrac{1}{\xi}) \right] \quad . \quad (B.14)$$

With the value of R from eq. (B.14), $(\exp ikR)/R$ is easily found to be

$$\frac{e^{ikR}}{R} \cong k \ \frac{e^{ic\xi}}{c\xi} \ e^{-ic \ \{\eta\eta_o\xi_s + (1-\eta^2)^{\frac{1}{2}} \ [(1-\eta_o^2) \ (\xi_s^2-1)]^{\frac{1}{2}} \ \cos \ \varphi_o\}} \ . \ (B.15)$$

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